

# Concatenated Codes for Amplitude Damping

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**Abstract**—We discuss a method to construct quantum codes correcting amplitude damping errors via code concatenation. The inner codes are chosen as asymmetric Calderbank-Shor-Steane (CSS) codes. By concatenating with outer codes correcting symmetric errors, many new codes with good parameters are found, which are better than the amplitude damping codes obtained by any previously known construction.

**Index Terms**—Quantum error-correcting codes, concatenated codes, amplitude damping channel, CSS codes.

## I. INTRODUCTION

Channels transmitting quantum information represented by the density matrix  $\rho$  are completely positive, trace-preserving linear maps. They can be represented in the Kraus decomposition  $\mathcal{A}(\rho) = \sum_k A_k \rho A_k^\dagger$  with  $\sum_k A_k^\dagger A_k = I$  [19]. The matrices  $A_i$  are called the Kraus operators or error set of the channel  $\mathcal{A}$ .

Most quantum error-correcting codes constructed so far are for the depolarizing channel

$$\mathcal{A}_{\text{DP}}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z), \quad (1)$$

where the Pauli  $X, Y, Z$  errors happen equally likely. The Kraus operators of  $\mathcal{A}_{\text{DP}}$  are  $\{\sqrt{1-p}I, \sqrt{\frac{p}{3}}X, \sqrt{\frac{p}{3}}Y, \sqrt{\frac{p}{3}}Z\}$ .

The assumption of equal probability for the Pauli  $X, Y, Z$  errors in fact models the worst case scenario of ‘white noise’, where all kind of errors happen. However, in practical systems, some errors are usually more likely to happen than others. A more realistic error model for physical systems is based on the common noise processes described by amplitude damping and phase damping. The corresponding asymmetric channel is given by

$$\mathcal{A}_{\text{AS}}(\rho) = (1 - (2p_{xy} + p_z))\rho + p_{xy}(X\rho X + Y\rho Y) + p_z Z\rho Z, \quad (2)$$

where the Pauli  $X$  and  $Y$  errors happen with equal probability  $p_{xy}$ , which is determined by the amplitude damping (AD) noise. The probability  $p_z$  of the Pauli  $Z$  error depends on the phase damping noise, and in general we have  $p_{xy} \neq p_z$ .

The amplitude damping channel is given by [19]

$$\mathcal{A}_{\text{AD}}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger, \quad (3)$$

where the Kraus operators are

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad (4)$$

where  $\gamma$  is a damping parameter. The AD channel models, e.g., photon loss in optical fibers, or spontaneous emission of atoms [2], [19].

It has been first demonstrated in [18] that designing QECCs adaptively to the AD noise can result in better codes. In particular, a four-qubit code correcting a single AD error was found, using less qubits than the smallest single-error-correcting code for the depolarizing channel that needs five qubits [1], [16]. Generalizations of this four-qubit code are discussed in [5], [6]. In particular, it was discussed in [6] that Shor’s nine-qubit code can correct 2 AD errors, despite the fact that the code only corrects a single error for the depolarizing channel.

Subsequent works borrow ideas from the construction of classical asymmetric codes [13], combined with the codeword stabilized (CWS) quantum code method [3], to construct single-error-correcting AD codes, including both stabilizer codes and non-additive codes [17], [21]. Multi-error-correcting AD codes are discussed in [4], based on a concatenation method. In particular, the inner code is chosen as the two-qubit code  $\{|01\rangle, |10\rangle\}$  based on the classical dual-rail code, which results in a quantum erasure channel for the outer codes. Many good stabilizer AD codes are constructed by concatenating with the quantum erasure codes. However, due to the choice of the inner code, the rate of the constructed AD codes can never exceed 1/2.

In this work, we discuss a new method to construct AD codes via concatenation. We choose the inner codes as codes correcting certain kind of asymmetric errors. By carefully analyzing the error model for the AD channel, we introduce the concept of ‘effective weight’ for errors and ‘effective distance’ for the AD codes. This allows us to use outer codes correcting symmetric errors (i.e., the ‘usual’ codes designed to correct depolarizing errors). Our new method results in many new AD codes with good parameters, which are better than the AD codes obtained by any previously known construction.

## II. BACKGROUND

A QECC  $Q$  is a subspace of the space of  $n$  qudits  $(\mathbb{C}^q)^{\otimes n}$ , with single qudit dimension  $q$ . For a  $K$ -dimensional code space spanned by the orthonormal basis  $\{|\psi_i\rangle, i = 1, \dots, K\}$  and an error set  $\mathcal{A}$ , there is a physical operation detecting all the elements  $A_\mu \in \mathcal{A}$  if the error detection condition [1], [14]

$$\langle \psi_i | A_\mu | \psi_j \rangle = c_\mu \delta_{ij} \quad (5)$$

is satisfied.

The notation  $((n, K))_q$  is used to denote a qudit QECC with length  $n$  and dimension  $K$ . A stabilizer QECC has dimension  $K = q^k$  for some integer  $k$ , and the notation  $[[n, k]]_q$  is used to denote a qudit stabilizer code with length  $n$  and dimension  $q^k$ . A code  $Q$  is of distance  $d$  if Eq. (5) is satisfied for all  $A_\mu$  that act nontrivially on at most  $d - 1$  qudits.

In this work, we focus on the construction of AD codes, which are qubit codes with  $q = 2$ . However, qudit codes with  $q = 2^r$  are used as outer codes for the concatenation constructions to get qubit AD codes.

We consider error sets  $\mathcal{A}$  of Pauli type. For multi-qubit Pauli operators, for instance,  $X \otimes Y \otimes I \otimes Z$ , we will write it as  $XYIZ$  or  $X_1Y_2Z_4$  (where the subscripts denote the number of the qubits that the Pauli  $X, Y, Z$  operator is acting on), when no confusion arises. For the AD channel  $\mathcal{A}_{\text{AD}}$  as given in Eq. (4), the Kraus operators  $A_0$  and  $A_1$  are not Pauli operators. However, we can find Pauli error models that lead to codes correcting AD errors.

Notice that

$$A_1 = \frac{\sqrt{\gamma}}{2}(X + iY), \quad A_0 = I - \frac{\gamma}{4}(I - Z) + O(\gamma^2), \quad (6)$$

and hence,  $A_0$  which is of different order in  $\gamma$  as  $A_1$ . So the corresponding asymmetric error model as given in Eq. (2) has  $p_{xy} \propto \gamma$  and  $p_z \propto \gamma^2$ .

A  $t$ -error-correcting AD code (or  $t$ -code in short) improves the fidelity of the transmitted state from  $1 - \gamma$  to  $1 - \gamma^t$ . For instance, for  $t = 1$ , we only need to correct a single  $A_1$  error and detect a single  $A_0$  error [10]. In terms of Pauli operators, we only need to correct a single  $X$  and  $Y$  error, and detect a single  $Z$  error. In other words, a code that detects the error set  $\mathcal{A}^{\{1\}}$  that is given by

$$\mathcal{A}^{\{1\}} = \{I\} \cup \{X_i, Y_i, Z_i, X_iX_j, X_iY_j, Y_iY_j\}, \quad (7)$$

with  $i, j \in [1, n]$ , is a 1-code that corrects a single AD error.

Pauli error models that lead to codes correcting  $t$  AD errors can be given similarly. For instance, codes detecting the Pauli error set given by  $\mathcal{A}^{\{2\}} = \{A_\mu A_\nu : A_\mu, A_\nu \in \mathcal{A}^{\{1\}}\}$  are 2-codes that correct 2 AD errors. We will similarly denote by  $\mathcal{A}^{\{t\}}$  the Pauli error set that results in  $t$ -codes.

### III. CONCATENATED METHOD

We examine the weight properties of the elements in  $\mathcal{A}^{\{t\}}$ , which leads to new effective error models that are more convenient for constructing codes detecting the error set  $\mathcal{A}^{\{t\}}$ . From Eq. (3) and Eq. (6) it follows that each  $Z$  error contributes a factor of  $\gamma^2$  to the noise, while  $X$  or  $Y$  errors contribute a factor of  $\gamma$ . In other words, when we consider each  $X$  or  $Y$  error as ‘1 error’, then each  $Z$  error will be ‘effectively 2 errors’. Motivated by this observation, we have the following definition for ‘effective weight’.

*Definition 1:* For any tensor product  $E$  of Pauli errors, each tensor factor  $X$  or  $Y$  has effective weight 1, and each factor  $Z$  has effective weight 2. The effective weight of  $E$  is the sum of the effective weight of all factors  $X, Y, Z$  in  $E$ , and is denoted by  $\text{wt}_e(E)$ .

As an example, for  $E = XYIZ \in \mathcal{A}^{\{2\}}$ ,  $\text{wt}_e(E) = 4$ . In fact, we have the following result on the effective weight of the elements in  $\mathcal{A}^{\{t\}}$ .

*Lemma 2:* Any element  $E \in \mathcal{A}^{\{t\}}$  has effective weight  $\text{wt}_e(E) \leq 2t$ .

*Proof:* Notice that any element  $E \in \mathcal{A}^{\{t\}}$  will be a product of at most  $t$  elements from  $\mathcal{A}^{\{1\}}$  as given in Eq. (7). Any element in  $\mathcal{A}^{\{1\}}$  has at most effective weight 2, hence  $E$  has at most effective weight  $2t$ . ■

Obviously, the upper bound  $2t$  is achievable by some elements  $E \in \mathcal{A}^{\{t\}}$ . We can now define the effective distance  $d_e$  for  $t$ -AD-error-correcting codes that detect the error set  $\mathcal{A}^{\{t\}}$ . This effective distance will later allow us to compare our new codes with codes for the depolarizing channel with the usual code distance  $d$  (i.e., each  $X, Y, Z$  has weight 1).

*Definition 3:* A code has effective distance  $d_e = s$ , if it detects Pauli errors of effective weight up to  $s - 1$ .

Therefore, if a code has effective distance  $d_e = 2t + 1$ , then it detects the error set  $\mathcal{A}^{\{t\}}$ , and is hence a  $t$ -code.

Now we are ready to present our concatenation method.

*Theorem 4:* Starting from an inner  $[[n_1, k_1]]_2$  code  $\mathcal{Q}_i$  with effective distance  $d_e$ , concatenation of an  $[[n_2, k_2, \delta]]_{2^{k_1}}$  qudit outer code  $\mathcal{Q}_o$  with distance  $\delta$  results in a concatenated code  $[[n_1n_2, k_1k_2]]_2$  with effective distance at least  $d_e\delta$ .

*Proof:* The concatenated code  $\mathcal{Q}$  is a stabilizer code with length  $n_1n_2$  and dimension  $(2^{k_1})^{k_2}$ , hence encoding  $k_1k_2$  qubits. Denote the stabilizer of  $\mathcal{Q}$  by  $S_{\mathcal{Q}}$ . It has two sets of generators. The first set is obtained by replacing each tensor factor of the generators of the stabilizer  $S_{\mathcal{Q}_o}$  of the outer code by the corresponding logical operator of the inner code. The second set is formed by the stabilizer  $S_i$  of the inner code acting on each block of  $n_1$  qubits.

For the outer code  $\mathcal{Q}_o$ , any nontrivial logical operator in  $C(S_{\mathcal{Q}_o}) \setminus S_{\mathcal{Q}_o}$  has weight at least  $\delta$ , where  $C(S)$  is the centralizer of the stabilizer  $S$ . Likewise, the logical operators in  $C(S_{\mathcal{Q}_i}) \setminus S_{\mathcal{Q}_i}$  of the inner code have effective weight at least  $d_e$ . The logical operators of the concatenated code are obtained by replacing each tensor factor in the logical operators of the outer code by the corresponding logical operator of the inner code. Those operators have effective weight at least  $d_e\delta$ . As for standard concatenation of quantum codes [15], multiplying a logical operator of  $\mathcal{Q}$  by an element of the stabilizer  $S_{\mathcal{Q}}$  will not result in an effective weight less than  $d_e\delta$ . ■

### IV. THE $[[r, r - 1]]_2$ INNER CODE

To examine the power of the construction for AD codes given in Theorem 4, we will start with simple inner codes. We take classical linear binary codes of distance 2 with length  $r$  and dimension  $r - 1$  (hence cardinality  $2^{r-1}$ ). For any length  $r$ , such a distance-2 code will be formed by all bit strings of length  $r$  with even Hamming weight. For any such classical code  $\mathcal{C}_r = [r, r - 1, 2]_2$ , the corresponding quantum code  $\mathcal{Q}_r = [[r, r - 1]]_2$  is spanned by the computational basis vectors  $|c_i\rangle$  for all  $c_i \in \mathcal{C}_r$ . We first examine the effective distance of  $\mathcal{Q}_r$ .

*Lemma 5:* The code  $\mathcal{Q}_r$  defined above has effective distance  $d_e = 2$ .

*Proof:* The only non-trivial element of the stabilizer  $S_r$  of the code  $\mathcal{Q}_r$  is the  $r$ -fold tensor product  $Z^{\otimes r}$ . We need to look at the effective weights of the logical operators that are in  $C(S_r) \setminus S_r$ , where  $C(S_r)$  is the centralizer of  $S_r$ . These are Pauli operators that commute with  $Z^{\otimes r}$ . Clearly, a single  $Z$  (i.e.,  $Z_i$ ) operator having effective weight two is in  $C(S_r) \setminus S_r$ , but this set does not contain a single  $X$  or  $Y$  operator. The tensor product of two  $X$  or  $Y$  operators (i.e.,  $X_i X_j, X_i Y_j, Y_i X_j, Y_i Y_j$ ) is in  $C(S_r) \setminus S_r$ . Therefore, every logical operator of  $\mathcal{Q}_r$  has effective weight at least two, and hence the effective distance of  $\mathcal{Q}_r$  is 2. ■

Since the dimension of the quantum code  $\mathcal{Q}_r$  is  $2^{r-1}$ , it can be used as inner code for the concatenation with a qudit outer codes with single qudit dimension  $q = 2^{r-1}$ . For the construction of a  $t$ -code, we need effective distance  $2t + 1$  for the concatenated code.

*Theorem 6:* Given an  $[[n, k, \delta]]_{2^{r-1}}$  stabilizer code, a quantum code  $\mathcal{Q}$  with parameters  $[[rn - 1, (r - 1)k]]_2$  and effective distance  $d_e \geq 2\delta - 1$  can be constructed. This is a  $t$ -code with  $t = \delta - 1$ .

*Proof:* We start from an  $[[n, k, \delta]]_{2^{r-1}}$  stabilizer code of length  $n$ , and each qudit has dimension  $2^{r-1}$ . The first qudit is encoded into a trivial qubit code with parameters  $[[r - 1, r - 1, d_e = 1]]_2$ . Each of the other qudits  $j = 2, 3, \dots, n$  is encoded into the code  $\mathcal{Q}_r$  with parameters  $[[r, r - 1, d_e = 2]]_2$ . The resulting concatenated code  $\mathcal{Q}$  is a stabilizer code of length  $(r - 1) + (n - 1)r = rn - 1$  and dimension  $(2^{r-1})^k$ , hence encoding  $(r - 1)k$  qubits. Any logical operator of  $[[n, k, \delta]]_{2^{r-1}}$  has weight at least  $\delta$ . Hence any logical operator of  $\mathcal{Q}$  that acts trivially on the first qudit has effective weight at least  $2\delta$ . Logical operators of  $\mathcal{Q}$  that act non-trivially on the first qudit have effective weight at least  $1 + 2(\delta - 1) = 2\delta - 1$ . Therefore, the effective distance of  $\mathcal{Q}$  is  $d_e \geq 2\delta - 1$ . ■

*Example 7:* Starting from the  $[[5, 1, 3]]_2$  code with stabilizer generated by

$$\begin{array}{cccc} X & Z & Z & X & I \\ I & X & Z & Z & X \\ X & I & X & Z & Z \\ Z & X & I & X & Z \end{array}$$

and encoding qubits 2, 3, 4, 5 into the code  $\mathcal{Q}_2$  stabilized by  $ZZ$ , we get a  $[[9, 1]]_2$  code with effective distance  $d = 2 \cdot 3 - 1 = 5$ , which corrects two AD errors. By choosing the logical operators for  $\mathcal{Q}_2$  as  $\bar{X} = XX$  and  $\bar{Z} = ZI$ , the stabilizer of the  $[[9, 1]]_2$  code is generated by

$$\begin{array}{ccccc} X & ZI & ZI & XX & II \\ I & XX & ZI & ZI & XX \\ X & II & XX & ZI & ZI \\ Z & XX & II & XX & ZI \\ I & ZZ & II & II & II \\ I & II & ZZ & II & II \\ I & II & II & ZZ & II \\ I & II & II & II & ZZ \end{array}$$

Notice the two groups of generators as mentioned in the proof of Theorem 4.

We remark that the  $[[9, 1]]_2$  2-code given above is in fact local Clifford equivalent to one of the  $[[9, 1]]_2$  codes found in [12] via exhaustive numerical search for CWS codes detecting the error set  $\mathcal{A}^{\{2\}}$ . It is one of the best 2-codes known, which beats the  $[[10, 1]]_2$  2-code found in [4]. In fact, the construction in [4] can be viewed as a special case of Theorem 6, by concatenating all qudits of an outer code with the inner code  $\mathcal{Q}_2$ . Notice that in [4], codes with effective distance  $2\delta$  are constructed in order to obtain  $t$ -codes with  $t = \delta - 1$ , which results in length  $2n$  instead of  $2n - 1$  as given by Theorem 6. In other words, by using Theorem 6, the length of any  $t$ -code constructed in [4] can be reduced by one.

For decoding, the inner code  $[[r, r - 1]]_2$  will be used to detect single  $X$ - and  $Y$ -errors. This provides side-information on detected errors (erasures) for the outer code and allows to simultaneously correct  $e$  erasures and  $f$  erroneous blocks with  $r$  qubits each, as long as  $e + 2f < \delta$ .

## V. PARAMETERS OF NEW AD CODES

In this section we discuss the parameters of the new AD codes found by our concatenated method when using the inner code  $\mathcal{Q}_r$ . We compare the effective distance  $d_e$  of the new codes constructed via our concatenated method to the distance  $d_{lb}$  of the best known stabilizer codes.

The best possible parameters for our concatenation technique are expected when the outer code is an optimal quantum code, and quantum MDS (QMDS) codes in particular. QMDS codes have parameters  $[[n, n + 2 - 2d, d]]_q$ , i.e., they attain the quantum Singleton bound  $k + 2d \leq n + 2$  [14], [20]. QMDS codes are known to exist for all  $n \leq q + 1$ , for  $n = q^2 - 1, q^2, q^2 + 1$  and some  $d \leq q + 1$ , as well as for many parameters  $n \leq q^2 + 1, d \leq q + 1$  [8]. In general it seems as if for a qudit QMDS code with qudit dimension  $q$  we have the bounds  $d \leq q + 1$ , and  $n \leq q^2 + 1$ , with the exception of codes  $[[4^m + 2, 2^m - 4, 4]]_{2^m}$  (see [9]).

In order to construct a  $t$ -code, we use QMDS codes  $[[n, n - 2t, t + 1]]_q$  where  $q = 2^{r-1} \geq t$  as outer code and the code  $\mathcal{Q}_r = [[r, r - 1]]_2$  as inner code, yielding a  $t$ -code of length  $rn - 1$  encoding  $(r - 1)(n - 2t) = rn - n - 2rt + 2t$  qubits.

The parameters of our codes based on the concatenation of QMDS codes and the code  $\mathcal{Q}_r$  are presented in Table I. The last column labeled  $d_{lb}$  lists the largest known lower bound  $d_{lb}$  on the minimum distance of a stabilizer code for the depolarizing channel (see [7]). Here we consider only codes of length up to  $n_{\max} = 128$ . We only list the parameters  $[[n, k, d_e = 2t + 1]]_2$  of  $t$ -codes for which the effective distance  $d_e$  exceeds the lower bound  $d_{lb}$  (i.e.,  $d_e > d_{lb}$ ). Furthermore, we omit parameters for which we find even better codes (smaller length, larger dimension, or larger effective distance).

In Tables II and III we list parameters of the best  $t$ -codes we found using outer codes that do not reach the quantum Singleton bound  $k + 2d \leq n + 2$ , but have the largest minimum distance among the known codes. The codes in Table II are based on qubit codes as outer codes and hence comparable to the codes in [4], but reducing the length by one as discussed above.

TABLE I  
CONCATENATED CODES  $[[n, k, d_e]]_2$  FOR THE AD CHANNEL BASED ON  
QMDS OUTER CODES WITH QUDIT DIMENSION 2, 4, 8, AND 16.

$t$	concatenated code	outer code	$d_{lb}$
1	$[[7, 2, d_e = 3]]_2$	$[[4, 2, 2]]_2$	2
2	$[[9, 1, d_e = 5]]_2$	$[[5, 1, 3]]_2$	3

$t$	concatenated code	outer code	$d_{lb}$
3	$[[23, 4, d_e = 7]]_2$	$[[8, 2, 4]]_{2^2}$	6
	$[[26, 6, d_e = 7]]_2$	$[[9, 3, 4]]_{2^2}$	6
	$[[29, 8, d_e = 7]]_2$	$[[10, 4, 4]]_{2^2}$	6
	$[[41, 16, d_e = 7]]_2$	$[[14, 8, 4]]_{2^2}$	6
4	$[[26, 2, d_e = 9]]_2$	$[[9, 1, 5]]_{2^2}$	8
	$[[50, 18, d_e = 9]]_2$	$[[17, 9, 5]]_{2^2}$	8

$t$	concatenated code	outer code	$d_{lb}$
4	$[[39, 6, d_e = 9]]_2$	$[[10, 2, 5]]_{2^3}$	8
	$[[43, 9, d_e = 9]]_2$	$[[11, 3, 5]]_{2^3}$	8
	$[[47, 12, d_e = 9]]_2$	$[[12, 4, 5]]_{2^3}$	8
	$[[59, 21, d_e = 9]]_2$	$[[15, 7, 5]]_{2^3}$	8
	$[[75, 33, d_e = 9]]_2$	$[[19, 11, 5]]_{2^3}$	8
5	$[[47, 6, d_e = 11]]_2$	$[[12, 2, 6]]_{2^3}$	10
	$[[63, 18, d_e = 11]]_2$	$[[16, 6, 6]]_{2^3}$	10
	$[[71, 24, d_e = 11]]_2$	$[[18, 8, 6]]_{2^3}$	10
	$[[75, 27, d_e = 11]]_2$	$[[19, 9, 6]]_{2^3}$	10
	$[[79, 30, d_e = 11]]_2$	$[[20, 10, 6]]_{2^3}$	9
	$[[83, 33, d_e = 11]]_2$	$[[21, 11, 6]]_{2^3}$	10
	$[[91, 39, d_e = 11]]_2$	$[[23, 13, 6]]_{2^3}$	10
	$[[99, 45, d_e = 11]]_2$	$[[25, 15, 6]]_{2^3}$	10
	$[[103, 48, d_e = 11]]_2$	$[[26, 16, 6]]_{2^3}$	10
	$[[107, 51, d_e = 11]]_2$	$[[27, 17, 6]]_{2^3}$	10
	$[[111, 54, d_e = 11]]_2$	$[[28, 18, 6]]_{2^3}$	10
6	$[[95, 36, d_e = 13]]_2$	$[[24, 12, 7]]_{2^3}$	12
	$[[99, 39, d_e = 13]]_2$	$[[25, 13, 7]]_{2^3}$	11
	$[[103, 42, d_e = 13]]_2$	$[[26, 14, 7]]_{2^3}$	11
	$[[107, 45, d_e = 13]]_2$	$[[27, 15, 7]]_{2^3}$	11
	$[[111, 48, d_e = 13]]_2$	$[[28, 16, 7]]_{2^3}$	11
	$[[115, 51, d_e = 13]]_2$	$[[29, 17, 7]]_{2^3}$	12
	$[[119, 54, d_e = 13]]_2$	$[[30, 18, 7]]_{2^3}$	12
	$[[123, 57, d_e = 13]]_2$	$[[31, 19, 7]]_{2^3}$	12
7	$[[127, 60, d_e = 13]]_2$	$[[32, 20, 7]]_{2^3}$	11
	$[[127, 54, d_e = 15]]_2$	$[[32, 18, 8]]_{2^3}$	13

$t$	concatenated code	outer code	$d_{lb}$
6	$[[79, 16, d_e = 13]]_2$	$[[16, 4, 7]]_{2^4}$	12
7	$[[119, 40, d_e = 15]]_2$	$[[24, 10, 8]]_{2^4}$	14

## VI. DISCUSSION

We can also use other asymmetric codes as inner codes to construct concatenated codes based on Theorem 4. Using a similar idea as in Theorem 6, one may also encode the first qudit of the outer  $[[n_2, k_2]]_{2^{k_2}}$  code into a trivial  $[[k_2, k_2]]_2$  code. This leads to the following corollary.

*Corollary 8:* Concatenating an  $[[n_2, k_2, \delta]]_{2^{k_1}}$  qudit outer code  $\mathcal{Q}_o$  with an inner asymmetric  $[[n_1, k_1]]_2$  code  $\mathcal{Q}_i$  with effective distance  $d_e$  results in a code  $[[n_1(n_2 - 1) + k_2, k_1 k_2]]_2$  with effective distance at least  $d_e(\delta - 1) + 1$ , as well as a concatenated code  $[[n_1 n_2, k_1 k_2]]_2$  with effective distance at least  $d_e \delta$ .

TABLE II  
CONCATENATED CODES  $[[n, k, d_e]]_2$  FOR THE AD CHANNEL BASED ON  
NON-QMDS OUTER QUBIT CODES.

$t$	concatenated code	outer code	$d_{lb}$
3	$[[19, 2, d_e = 7]]_2$	$[[10, 2, 4]]_2$	6
	$[[23, 4, d_e = 7]]_2$	$[[12, 4, 4]]_2$	6
4	$[[21, 1, d_e = 9]]_2$	$[[11, 1, 5]]_2$	7
	$[[31, 4, d_e = 9]]_2$	$[[16, 4, 5]]_2$	8
	$[[35, 6, d_e = 9]]_2$	$[[18, 6, 5]]_2$	8
5	$[[31, 2, d_e = 11]]_2$	$[[16, 2, 6]]_2$	10
	$[[39, 4, d_e = 11]]_2$	$[[20, 4, 6]]_2$	9
	$[[41, 5, d_e = 11]]_2$	$[[21, 5, 6]]_2$	9
	$[[47, 6, d_e = 11]]_2$	$[[24, 6, 6]]_2$	10
	$[[55, 12, d_e = 11]]_2$	$[[28, 12, 6]]_2$	10
6	$[[33, 1, d_e = 13]]_2$	$[[17, 1, 7]]_2$	11
	$[[47, 3, d_e = 13]]_2$	$[[24, 3, 7]]_2$	11
	$[[49, 5, d_e = 13]]_2$	$[[25, 5, 7]]_2$	11
	$[[59, 8, d_e = 13]]_2$	$[[30, 8, 7]]_2$	12
	$[[63, 10, d_e = 13]]_2$	$[[32, 10, 7]]_2$	12
7	$[[47, 1, d_e = 15]]_2$	$[[24, 1, 8]]_2$	13
	$[[51, 4, d_e = 15]]_2$	$[[26, 4, 8]]_2$	12
	$[[59, 5, d_e = 15]]_2$	$[[30, 5, 8]]_2$	13
	$[[63, 6, d_e = 15]]_2$	$[[32, 6, 8]]_2$	14
	$[[65, 7, d_e = 15]]_2$	$[[33, 7, 8]]_2$	13
	$[[67, 8, d_e = 15]]_2$	$[[34, 8, 8]]_2$	14
	$[[71, 12, d_e = 15]]_2$	$[[36, 12, 8]]_2$	14
8	$[[49, 1, d_e = 17]]_2$	$[[25, 1, 9]]_2$	13
	$[[53, 3, d_e = 17]]_2$	$[[27, 3, 9]]_2$	13
	$[[69, 4, d_e = 17]]_2$	$[[35, 4, 9]]_2$	15
	$[[101, 19, d_e = 17]]_2$	$[[51, 19, 9]]_2$	16
9	$[[55, 2, d_e = 19]]_2$	$[[28, 2, 10]]_2$	14
	$[[71, 3, d_e = 19]]_2$	$[[36, 3, 10]]_2$	15
	$[[105, 17, d_e = 19]]_2$	$[[53, 17, 10]]_2$	17
10	$[[57, 1, d_e = 21]]_2$	$[[29, 1, 11]]_2$	15
	$[[81, 3, d_e = 21]]_2$	$[[41, 3, 11]]_2$	18
	$[[95, 4, d_e = 21]]_2$	$[[48, 4, 11]]_2$	20
	$[[97, 5, d_e = 21]]_2$	$[[49, 5, 11]]_2$	19
11	$[[83, 2, d_e = 23]]_2$	$[[42, 2, 12]]_2$	19
	$[[97, 3, d_e = 23]]_2$	$[[49, 3, 12]]_2$	21
	$[[99, 4, d_e = 23]]_2$	$[[50, 4, 12]]_2$	20
	$[[107, 8, d_e = 23]]_2$	$[[54, 8, 12]]_2$	19
12	$[[85, 1, d_e = 25]]_2$	$[[43, 1, 13]]_2$	21
	$[[101, 3, d_e = 25]]_2$	$[[51, 3, 13]]_2$	21
	$[[113, 5, d_e = 25]]_2$	$[[57, 5, 13]]_2$	21
13	$[[103, 2, d_e = 27]]_2$	$[[52, 2, 14]]_2$	21
	$[[115, 4, d_e = 27]]_2$	$[[58, 4, 14]]_2$	22
	$[[125, 6, d_e = 27]]_2$	$[[63, 6, 14]]_2$	23
14	$[[105, 1, d_e = 29]]_2$	$[[53, 1, 15]]_2$	21
	$[[117, 3, d_e = 29]]_2$	$[[59, 3, 15]]_2$	23
15	$[[119, 2, d_e = 31]]_2$	$[[60, 2, 16]]_2$	23
16	$[[121, 1, d_e = 33]]_2$	$[[61, 1, 17]]_2$	25

*Example 9:* Choose the inner code to be the asymmetric  $[[8, 3, \{4, 2\}]]_2$  CSS code with  $X$ -distance  $d_X = 4$  and  $Z$ -distance  $d_Z = 2$ , resulting in effective distance  $d_e = 4$ . It can be constructed from the first order Reed-Muller code and the repetition code. Its stabilizer is generated by

$$\begin{array}{cccccccc}
 Z & Z & Z & Z & I & I & I & I \\
 Z & Z & I & I & Z & Z & I & I \\
 Z & I & Z & I & Z & I & Z & I \\
 Z & Z & Z & Z & Z & Z & Z & Z \\
 X & X & X & X & X & X & X & X
 \end{array}$$



TABLE III  
CONCATENATED CODES  $[[n, k, d_e]]_2$  FOR THE AD CHANNEL BASED ON  
NON-QMDS OUTER CODES WITH QUDIT DIMENSION 4 AND 8.

$t$	concatenated code	outer code	$d_b$
4	$[[41, 8, d_e = 9]]_2$	$[[14, 4, 5]]_{22}$	8
	$[[44, 10, d_e = 9]]_2$	$[[15, 5, 5]]_{22}$	8
	$[[47, 12, d_e = 9]]_2$	$[[16, 6, 5]]_{22}$	8
5	$[[50, 10, d_e = 11]]_2$	$[[17, 5, 6]]_{22}$	9
6	$[[44, 2, d_e = 13]]_2$	$[[15, 1, 7]]_{22}$	12
	$[[56, 6, d_e = 13]]_2$	$[[19, 3, 7]]_{22}$	12
	$[[59, 8, d_e = 13]]_2$	$[[20, 4, 7]]_{22}$	12
	$[[74, 14, d_e = 13]]_2$	$[[25, 7, 7]]_{22}$	12
	$[[77, 16, d_e = 13]]_2$	$[[26, 8, 7]]_{22}$	12
	$[[80, 18, d_e = 13]]_2$	$[[27, 9, 7]]_{22}$	12
7	$[[104, 26, d_e = 15]]_2$	$[[35, 13, 8]]_{22}$	14
	$[[107, 28, d_e = 15]]_2$	$[[36, 14, 8]]_{22}$	14
8	$[[74, 6, d_e = 17]]_2$	$[[25, 3, 9]]_{22}$	15
	$[[92, 14, d_e = 17]]_2$	$[[31, 7, 9]]_{22}$	16
	$[[95, 16, d_e = 17]]_2$	$[[32, 8, 9]]_{22}$	16
	$[[110, 22, d_e = 17]]_2$	$[[37, 11, 9]]_{22}$	16
9	$[[77, 4, d_e = 19]]_2$	$[[26, 2, 10]]_{22}$	16
	$[[98, 10, d_e = 19]]_2$	$[[33, 5, 10]]_{22}$	18
	$[[101, 12, d_e = 19]]_2$	$[[34, 6, 10]]_{22}$	17
	$[[104, 14, d_e = 19]]_2$	$[[35, 7, 10]]_{22}$	17
	$[[110, 18, d_e = 19]]_2$	$[[37, 9, 10]]_{22}$	17
	$[[113, 20, d_e = 19]]_2$	$[[38, 10, 10]]_{22}$	18
	$[[116, 22, d_e = 19]]_2$	$[[39, 11, 10]]_{22}$	18
10	$[[95, 4, d_e = 21]]_2$	$[[32, 2, 11]]_{22}$	20
	$[[98, 6, d_e = 21]]_2$	$[[33, 3, 11]]_{22}$	19
	$[[101, 8, d_e = 21]]_2$	$[[34, 4, 11]]_{22}$	19
	$[[104, 10, d_e = 21]]_2$	$[[35, 5, 11]]_{22}$	18
	$[[107, 12, d_e = 21]]_2$	$[[36, 6, 11]]_{22}$	18
	$[[116, 14, d_e = 21]]_2$	$[[39, 7, 11]]_{22}$	20
	$[[119, 16, d_e = 21]]_2$	$[[40, 8, 11]]_{22}$	20
	$[[122, 18, d_e = 21]]_2$	$[[41, 9, 11]]_{22}$	20
	$[[125, 20, d_e = 21]]_2$	$[[42, 10, 11]]_{22}$	20
	$[[128, 22, d_e = 21]]_2$	$[[43, 11, 11]]_{22}$	20
11	$[[116, 10, d_e = 23]]_2$	$[[39, 5, 12]]_{22}$	21
	$[[119, 12, d_e = 23]]_2$	$[[40, 6, 12]]_{22}$	21
	$[[122, 14, d_e = 23]]_2$	$[[41, 7, 12]]_{22}$	21
	$[[125, 16, d_e = 23]]_2$	$[[42, 8, 12]]_{22}$	21
12	$[[116, 6, d_e = 25]]_2$	$[[39, 3, 13]]_{22}$	22
	$[[119, 8, d_e = 25]]_2$	$[[40, 4, 13]]_{22}$	22
	$[[128, 10, d_e = 25]]_2$	$[[43, 5, 13]]_{22}$	23

$t$	concatenated code	outer code	$d_b$
8	$[[107, 21, d_e = 17]]_2$	$[[27, 7, 9]]_{23}$	16

Based on Theorem 4, concatenating with a QMDS  $[[10, 2, 5]]_{23}$  outer code results in a code  $[[80, 6]]_2$  with effective distance  $d_e = 20$ . This code is better than the best known stabilizer code  $[[80, 6, 16]]_2$ . Using Corollary 8, we get a  $[[75, 6]]_2$  code with effective distance  $d_e = 17$ , correcting  $t = 8$  AD errors. This again improves upon the best known stabilizer code  $[[75, 6, 15]]_2$ . However, the  $t = 8$  code with parameters  $[[74, 6]]_2$  listed in Table III has better parameters. Note that for both codes  $[[8, 3, \{4, 2\}]]_2$  and  $[[2, 1, \{2, 1\}]]_2$  (i.e., the code  $Q_2$  with the stabilizer generated by  $ZZ$ ), the ratio between the  $X$ - and  $Z$ -distance is 2, resulting in an effective distance of 4 and 2, respectively. However, the  $[[2, 1, \{2, 1\}]]_2$  code has rate  $1/2$

compared to rate  $3/8$  for the  $[[8, 3, \{4, 2\}]]_2$  code, resulting in codes with better parameters.

Nonetheless, this example illustrates the flexibility of our method. We can also use it for channels for which the asymmetry between  $p_{xy}$  and  $p_z$  is different than for the amplitude damping channel (see, e.g. [11]).

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